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$1/f$ noise and spectral singularities in strongly disordered electronic systems

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Abstract. This paper discusses the common origin of $1/f$ noise in the power spectrum of random walks in a random environment and the $1/E$-like spectral anomaly for tight-binding electronic systems with off-diagonal disorder. In one dimension the existence of the Dyson singularity, $\langle\rho(E)\rangle \propto |\ln^{-3}|E||/|E|$, for the density of states of the random chain as $|E| \to 0$ is inevitably linked to Sinai ultraslow diffusion with the law $\langle x^2(t)\rangle \propto \ln^4 t$ as $t \to 0$ and the presence of $S(f) \propto \ln^4 f/f$ power spectral density for small frequencies f. Different forms of power-law singularities are expected instead for a correlated disorder model appropriate to describe random symmetric diffusion and magnon or phonon excitations. The two problems are discussed in terms of the averaged moments of the electronic wavefunction. The $1/E$ behaviour is shown to rely on the underlying very general validity of the log–normal distribution for strongly disordered electronic systems. Analytical arguments are given and numerical evidence reported for the asymptotic presence of these singularities in the dimensions $d = 2, 3$ when the disorder is sufficiently strong.

1. Introduction

Much work has been devoted to the study of quantum transport in disordered systems in connection with the phenomenon of Anderson localisation [1]. The related subject of classical excitation dynamics which is concerned with wave propagation (classical diffusion, magnons, phonons) in random media has also received much attention. In fact, the existence of long-time classical diffusion and the presence of extended long-wavelength Goldstone modes in random magnets seems to contradict the well known statements about the absence of quantum diffusion in random chains [2]. An answer to this problem has already been given in [3] where it is pointed out that the classical problem can be mapped exactly onto a quantum electronic model realised within a tight-binding Hamiltonian with off-diagonal disorder. The symmetric classical diffusion maps onto a chain with a particular form of correlated off-diagonal disorder where the random bonds occur in pairs. In this case the known rules about the absence of extended states in one dimension in the presence of a random potential do not apply. The electronic structure near the band centre becomes indistinguishable from that of a pure chain and extended states appear as $|E| \to 0$ in accordance with the presence of normal diffusion at long times. A distinctly different behaviour is expected for classical diffusion with asymmetric hopping rates in a random environment, which has also been studied. In the marginally asymmetric case a model known as random–random walk can be defined by a stochastic equation with an additional randomly distributed time-independent drift force [4]. The dynamics is drastically different from normal diffusion. It is dominated by untypically long times taken by the particle to cross 'mountains' and the diffusion becomes logarithmically slow at long times as has been proved by Sinai [5]. Moreover, the power spectral density for the distance autocorrelation function exhibits $1/f$ behaviour at small frequencies, f, a phenomenon known as $1/f$ noise [6]. The corresponding one-dimensional quantum problem is the Anderson tight-binding Hamiltonian with off-diagonal (non-correlated) disorder and the Dyson $1/|E|$ singularity, within logarithmic corrections, is expected to occur near the band centre. The average localisation length also diverges logarithmically at this energy but the $E = 0$ state is localised in contrast to the long-wavelength magnon states. The purpose of this paper is to discuss the quantum disordered problem bearing in mind the results for its classical analogue. In fact, the Dyson singularity will be connected with the presence of $1/f$ noise and the results extended to the quantum problem in higher dimensions. On the basis of a qualitative theory the presence of the Dyson singularity is shown in any dimension only when the disorder is sufficiently strong. Numerical simulation results, via two methods, are presented which tend to favour this conclusion.

The $1/f$ noise or flicker noise is a very common phenomenon in many areas of physics [6]. It has been observed in many other situations as well, such as fluctuations in oceanic currents, loudness of music, etc. Due to its appearance in so many conceptually different cases it is believed that there is no unique theoretical explanation. The majority of the proposed mechanisms of explanation agree that it is an equilibrium phenomenon [7], a belief which does not contradict the experiments. It has further been suggested [7] that in order to achieve an explanation two ingredients are required: a power-law form for the spectral correlation functions and the presence of self-similarity. Our case falls under the statement that disorder is a general mechanism responsible for the phenomenon [4]. The random ensemble under study has many untypical configurations which eventually dominate over the calculated mean values. From a mathematical point of view the random process is often expressed as a product of many random variables and the log–normal distribution law is recovered. It can easily be shown that the log–normal distribution with a sufficiently large variance generates $1/f$ noise for a range of frequencies f depending on the magnitude of the variance. Log–normal distributions are believed to describe the sample-to-sample fluctuations of many local quantities (current, wavefunction amplitude, etc) in strongly disordered electronic systems. In fact recent results [8] obtained from the non-linear sigma in the $2 + \varepsilon$ dimensions model indicate the asymptotic presence of the log–normal distribution in the localised regime in any dimension. It was argued that the distributions should have instead only log–normal tails in the metallic regime. In the latter case the variance of the conductance appears as a universal number. In [9] a one-dimensional model is introduced to study the energy width distributions for the localised states. Subsequently the occupation fluctuation spectrum is deduced and the $1/f$ noise discussed [10]. The logarithmic classical diffusion has been connected to the low-energy properties of a quantum model before [11]. In this paper an alternative viewpoint is presented to discuss the $1/f$ noise in insulators. It is the log–normal distribution which is important and the off-diagonal disorder allows the phenomenon to manifest itself. In one dimension the mapping of the quantum problem to the classical diffusion model makes the display of $1/f$ noise in the form of a Dyson singularity unambiguous. Moreover, our numerical results for higher dimensions may be viewed as a confirmation for the use of a one-dimensional model [9], asymptotically, in any dimension as long as the disorder is very strong.

The paper is organised as follows: in section 2 we discuss the quantum disordered system and its isomorphic classical analogue in one dimension. We exploit the existence of an exact mapping between the two problems which defines relationships between the long-time classical properties and the small-$|E|$ quantum behaviour [3, 11]. The two cases of off-diagonal disorder are distinguished. We introduce the family of generalised Lyapunov exponents [12, 13] and discuss the basic quantities of interest near the band centre. The differences between the two models are clarified. In section 3 we present our approach for tackling higher-dimensional quantum problems. A suitable tri-diagonalisation transformation of a higher-dimensional problem to a semi-infinite one-dimensional chain is outlined. The mapping is exact but the chain obtained is inhomogeneously disordered, that is the variance of the bond randomness decays along the chain. We can test our expectation that strong disorder maps onto an ordinary disordered chain (i.e. with constant variance along the chain) so that the Dyson singular behaviour is eventually recovered. We present numerical results for $d = 2, 3$ which indicate the limits of such behaviour. Results for the density of states for weak disorder are also shown, for a comparison with a previous qualitative theory [14], large-n expansions [15] and numerical results [16]. Finally we summarise our approach and discuss our results in connection with other sources of $1/f$ noise.

2. One dimension

Firstly, we discuss the electronic tight-binding Hamiltonian with off-diagonal disorder in a one-dimensional bipartite lattice. The stationary difference equation is

$$V_{n-1}\Psi_{n-1} + V_n\Psi_{n+1} = E\Psi_n \tag{1}$$

where n covers all lattice sites. E is the energy eigenvalue, Ψ_n denotes the wavefunction component at the nth lattice site and V_n are the independent random off-diagonal matrix elements. For the case of uncorrelated disorder $\ln V_n$ is independently distributed with zero mean and given variance σ^2. We consider first the case of correlated disorder. This is often referred to as the case with spin wave symmetry [3], the V_n are random but they occur in pairs, that is $V_{2n} = V_{2n+1}$. Then if we eliminate odd sites in equation (1) we arrive at a new equation for the even sites, which is

$$V^2_{2(n-1)}\Psi_{2(n-1)} + V^2_{2n}\Psi_{2(n+1)} = (E^2 - V^2_{2(n-1)} - V^2_{2n})\Psi_{2n}. \tag{2}$$

Equation (2) reduces to the basic amplitude equation describing a single magnon in a linear ferromagnetic Heisenberg chain if $\varepsilon = E^2$, $J_n = V^2_{2n}$, $\tilde{\Psi}_n = \Psi_{2n}$ then

$$J_{n-1}\tilde{\Psi}_{n-1} + J_n\tilde{\Psi}_{n+1} = (\varepsilon - J_{n-1} - J_n)\tilde{\Psi}_n \tag{3}$$

where J_n denotes the exchange interaction between spins at sites n and $n + 1$. Therefore the long wavelength behaviour of the excitations should be exactly identified from the correlated disorder model near the band centre. Equation (3) may be viewed as the Laplace transform of a corresponding Master equation describing classical diffusion with symmetric probabilities (equal to $\frac{1}{2}$) for left and right moves. For the random–random-walk problem the probabilities for left and right movements are random numbers chosen from a flat distribution in $(0, 1)$. Its quantum analogue is described by equation (1) with a choice of independent, random and uncorrelated V_n. The classical discrete time model is described by

$$p_{n-1}P_{n-1}(t) + q_{n+1}P_{n+1}(t) = P_n(t+1) \tag{4}$$

where $P_n(t)$ is the probability that the particle is on site n at time t and p_n, $q_n = 1 - p_n$ are the hopping probabilities to sites $n+1$ and $n-1$ with $\langle \ln(p_n/q_n)\rangle = 0$. The equation

$$\mathrm{d}P_n/\mathrm{d}t = T_{n,n-1}P_{n-1} + T_{n,n+1}P_{n+1} - (T_{n-1,n} + T_{n+1,n})P_n \tag{5}$$

describes a continuous time hopping model instead. P_n is the probability for being at site n, and T_{nm} and T_{mn} are the probabilities for the transitions $n \to m$ and $m \to n$, respectively. It can be shown that equation (5) is isomorphic to the Anderson model under certain conditions [11].

The averaged density of states and the localisation length near the band centre were obtained in [3] by a combination of perturbation theory and a scaling assumption. These results permitted a clear difference for the two cases of disorder to be established. For the non-correlated model the average density of states diverges as

$$\langle \rho(E)\rangle \propto \sigma^2 |\ln^{-3}|E||/|E| \qquad \text{as } |E| \to 0, \tag{6}$$

which is the well known Dyson result [17]. For the correlated model (equation (3)) the dependence on the energy is identical to that of the pure chain if the disorder is not too strong. This behaviour explains the pure long-wavelength magnon density of states ($\langle \rho(\varepsilon)\rangle \propto \varepsilon^{-1/2}$) often observed in low-dimensional random magnets. The pure dispersion law $\varepsilon \propto Dk^2$ is satisfied, where D is the spin wave stiffness which is proportional to $\langle 1/J\rangle^{-1}$ (see [3]). In the case of very strong disorder, for example, when $\langle 1/J\rangle = \infty$, the corresponding behaviour is described by non-universal power laws which depend on the choice of distribution for the bond strengths [3]. For the electronic problem the localisation length diverges at $|E| = 0$ as a power law and logarithmically for the correlated and non-correlated models, respectively. However, extended states may appear in the band centre only in the former case while in the latter a typical wavefunction decays as $\exp(-\text{constant} \times \sqrt{N})$ instead. Interesting analogies exist between this result and the first-passage time in random–random walks [18].

In order to make transparent the differences in the localisation properties of the two electronic models we introduce the family of localisation lengths as suggested in [12]:

$$\xi_q^{-1} = \lim_{n\to\infty} (1/nq) \ln\langle |\Psi_n|^{-q}\rangle \qquad \forall q \neq 0 \tag{7}$$

where

$$\gamma = \xi_0^{-1} = \lim_{n\to\infty} (1/n)\langle \ln|\Psi_n|\rangle$$

is the usual inverse localisation length. At $E = 0$ the transfer matrices derived from equation (1) commute and the problem is easy. If we assume that $|\Psi_0| = 1$ we may estimate a quantity of interest which is the response at the $2N$th site. In fact, we want to study its distribution properties. From equation (1) we obtain:

$$\ln|\Psi_{2N}| = \sum_{n=1}^{N} \ln(V_{2n-2}/V_{2n-1}). \tag{8}$$

In the correlated case the summation in equation (8) is trivially zero, all $\xi_q^{-1} = 0$ and the state is extended. In the uncorrelated case if $\ln V_n$ is normally distributed with zero mean and σ^2 variance the sum in equation (8) has zero mean and $2N\sigma^2$ variance. This leads to $\gamma = 0$. If we make the cumulant expansion

$$\langle \ln|\Psi_{2N}|^q\rangle = q\langle \ln|\Psi_{2N}|\rangle + (q^2/2)[\langle(\ln|\Psi_{2N}|)^2\rangle - \langle \ln|\Psi_{2N}|\rangle^2] + O(q^3) \tag{9}$$

we can easily evaluate the rest of the localisation lengths. We obtain a linear relation on q for the growth rate for the moments of $|\Psi_{2N}|$, that is

$$\xi_q^{-1} = (\sigma^2/2)q \tag{10}$$

due to the Gaussian distribution of $\ln|\Psi_{2N}|$ which makes only the q^2 term survive in the expansion (9). In this case we have for the generalised Lyapunov exponent $L(q) = q\xi_q^{-1}$:

$$L(q) = \gamma q + (\mu/2)q^2 \tag{11}$$

that is a parabolic law, with $\gamma = 0$ and $\mu = \sigma^2$. The γ and μ denote the mean and the variance for the asymptotic value of $(1/2N)\ln|\Psi_{2N}|$. In terms of the multifractal theory [13] it appears convenient to use, instead of $L(q)$, its Legendre transform $h(\alpha)$. From equation (11) $h(\alpha)$ takes again the simple parabolic form

$$h(\alpha) = L(1) - (1/2\sigma^2)\alpha^2 \tag{12}$$

with α ranging from $-\infty$ to $+\infty$, $h(\pm\sigma^2) = 0$ and the maximum is $h(0) = \frac{1}{2}\sigma^2$. We can observe that although γ is identically zero the state is not extended because the rest of ξ_q^{-1}, for all $q \neq 0$, are different from zero. A hierarchy of (related in this case) decay properties for the wavefunction exists which is similar to the multifractal singularities. A typical decay of the wavefunction amplitude is slower than exponential and the state is weakly localised. This result should be compared with super-localisation [1], where a decay faster than exponential occurs and $\xi_q^{-1} = \infty$ for all q.

We now return to the case of $E \neq 0$ where the transfer matrices do not commute. The results of [4] for the non-correlated case give for the Lyapunov exponent for small $|E|$

$$\gamma \propto \sigma^2/|\ln(1/E^2)| \tag{13}$$

and for $\langle\rho(E)\rangle$ the Dyson singularity of equation (6) [17]. The ξ_q^{-1} should also exhibit similar scaling laws as a function of E and σ^2. Equations (13) and (6) were numerically verified for all kinds of disorder in [19]. We have also investigated numerically the value of μ from equation (11). We find a linear dependence of μ on σ^2 as at $|E| = 0$, while μ is almost independent of $|E|$. It is highly improbable that this case will differ from the normal expectation that for one-dimensional disordered systems, at least for a bounded disordered potential, the log–normal distribution is appropriate to describe local quantities, including $|\Psi_{2N}|$ [8]. Therefore the non-Gaussian fluctuation effects for the log observed in [13] for weak disorder in one dimension are not generally met here. They imply higher-order terms in equation (9) and deviations from log–normal behaviour with a set of many unrelated scaling exponents. It should be stressed that our results concern the strongly localised regime and the limit of infinite size is taken. It is therefore obvious that a finite-variance μ or even the non-Gaussian terms in equation (9) will not affect the factor $1/\sqrt{N}$ for the relative mean square root variance $\delta\gamma/\gamma$ and the localisation length is a self-averaging quantity in one dimension as expected from general theorems [1]. Of course, this is no longer true when the system size N becomes comparable or less than ξ_0. The mesoscopic fluctuations become important in this regime and $\delta\gamma/\gamma$ is of order one [8].

We have also considered the weakly disordered case, where the Dyson singularity should not appear until the energy is very close to zero. It is possible to obtain upper bounds for γ, which also serve as rough estimates, by using the fact that $\langle \ln x \rangle \leqslant \ln \langle x \rangle$. This implies that $\gamma \leqslant \frac{1}{2} \ln \Lambda_{\max}$, where $\Lambda_{\max}$ is the largest, in absolute value, eigenvalue of the averaged 4×4 direct product of the transfer matrices. We obtain for uncorrelated weak off-diagonal disorder $\gamma \leqslant \sigma^2 (1 + \frac{1}{8}E^2)$ which can be compared with the known result for weak diagonal disorder σ_D^2 [20]. At the band centre this defines the scaling relationship $\sigma^2 = \frac{1}{4}\sigma_D^2$ between off-diagonal and diagonal disorder which holds generally.

3. Higher dimensions, results for $d = 2, 3$

The $1/f$ spectrum of fluctuations should persist in two and higher dimensions only if logarithmic diffusion occurs. However, it is believed that this is the case only when the random potential, to whom the random force is a gradient, has long-range correlations. A random–random-walk model was introduced in [4] in the case where the random force is not a gradient but has short-range correlation. A square lattice was considered in two dimensions and the transition probability from a site to its nearest neighbours was taken as the Kth power of a random number chosen from a flat probability distribution on the interval (0, 1). Each probability was subsequently normalised so that the sum of probabilities starting from a given site was one. K measures the degree of correlation, for example, if $K = 0$ all probabilities are equal to a quarter for the square lattice and the standard (symmetric) random walk is obtained. Then the power spectral density is $S(f) \propto 1/f^2$ and the particle diffuses freely. $K = \infty$ leads to trapping. In this case the noise becomes white ($S(f)$ is constant), that is uncorrelated from site to site. For intermediate values of K a variety of different behaviours was obtained [4] at long times ranging from power-law diffusion, for small K, to logarithmic Sinai diffusion for larger values of K. In the latter case colouring of the noise is observed and $1/f$ noise in two dimensions. Although we cannot exactly connect the behaviour of the classical excitations and diffusion to a specific tight-binding Hamiltonian, as we have done in one dimension, the basic analogies between classical diffusion at long times and the low-$|E|$ behaviour for the quantum problem should not be affected. In the rest of the paper we consider the higher-dimensional quantum analogue of the classical problem, which is a tight-binding Hamiltonian with non-correlated off-diagonal disorder in $d = 2, 3$

The quantum Hamiltonian with off-diagonal disorder is given by

$$H = \sum_{(n,n')} V_{nn'} |n\rangle\langle n'| \tag{14}$$

where the $V_{nn'}$ are random and the sum extends to all nearest-neighbour pairs of lattice sites (n, n') in a d-dimensional hypercubic lattice. Due to the two-sublattice structure of the Hamiltonian of equation (14) the averaged density of states is symmetric around $E = 0$, that is $\rho(E) = \rho(-E)$. In the absence of disorder the well known Van Hove logarithmic singularity exists in $d = 2$ and no singularity is present in $d = 3$ [21]. For the disordered problem the singular behaviour which was found in one dimension persists, at least in two dimensions. For weak off-diagonal disorder in two dimensions numerical evidence for a sharper than log power-law singularity, that is

$$\langle \rho(E) \rangle \propto |E|^{-\varphi} \qquad \text{as} \quad |E| \to 0 \tag{15}$$

was obtained [19]. The numerical value for the exponent was $\varphi = 0.31 \pm 0.02$ [19]. This should be contrasted with the power law $\langle x^2(t)\rangle \propto t^{2\zeta}$ as $t\to\infty$, with $\zeta = 0.30 \pm 0.09$, obtained for the two-dimensional classical problem for not too large values of K. It is no longer possible to relate exactly the obtained subdiffusive behaviour to the magnon density of states as $\varepsilon \to 0$. However, it is probable that the nearly equal estimates for the exponents φ or ζ may also describe the low-ε magnon density of states in two dimensions [22]. In the case of very strong off-diagonal disorder the singularity was of a different nature and approached $1/|E|$ [19] in agreement with the logarithmic diffusion [4] obtained for large K. It is argued that the Dyson singularity should be recovered asymptotically in two dimensions and the desired $1/f$ behaviour obtained. For cubic lattices no singularity is believed to exist for weak disorder [14–16]. However, for very strong off-diagonal disorder we again expect the $1/|E|$ behaviour.

In order to discuss the two- and three-dimensional lattices the tridiagonalisation scheme is proposed, of reducing a Hamiltonian describing a higher than one-dimensional lattice to a semi-infinite linear chain [14]. This can generally be achieved for any Hermitian operator H defined on a Hilbert space which consists of the site basis set. Given a normalised starting vector $|1\rangle \in \mathbb{Z}^d$ for a particle at a given site operating successively on $|1\rangle$ with H, we construct a new sequence of orthogonalised vectors $|n\rangle$, $n = 2$, to N via the recursion relation

$$b_n|n\rangle = (H - a_n)\,|n-1\rangle - b_{n-1}|n-2\rangle. \tag{16}$$

The $|n\rangle$ are linear combinations of the original basis set and constitute a representation of H in the new basis set. They define a tridiagonal matrix where a_n are the diagonal and b_n the off-diagonal matrix elements. It should be stated that the order of the tridiagonal matrix N counts the number of shells in the lattice starting from site $|1\rangle$. This method is particularly suitable for the evaluation of averaged local densities of states. For the regular d-dimensional lattice $a_n = 0$ for all n and the b_n converge to d for $n \gg 1$. For our H (equation (14)), which has only off-diagonal matrix elements $a_n = 0$ for all n. Therefore, a Hamiltonian with off-diagonal disorder in any dimension should map exactly onto a semi-infinite chain with random hopping elements b_n. Equivalently this would imply that the one-dimensional results should carry through to higher dimensions. However, the chain obtained is not homogeneously disordered, and the b_n are random variables with variance σ_n^2 decaying along the chain as $n^{-2\alpha}$ with $\alpha > 0$. If $\alpha = 0$ the disorder is homogeneous and the well known one-dimensional results should remain valid in any dimension. We argue that this is the case for very strongly disordered systems in any dimension. But even if α is non-zero the higher-dimensional system can still be considered from the study of a specific one-dimensional model with decreasing disorder. This has already been done [1, 23] for a related model with identically distributed random-site energies with decaying variance described by an exponent α. The case of $\alpha = \frac{1}{2}$ (i.e. $\sigma_n^2 \propto n^{-1}$) is related to models of localisation in the presence of an electric field [1] where a transition from power-law localisation to a singular continuous spectrum was shown to occur for a critical value of the strength of the b_n. For $\alpha > \frac{1}{2}$ extended states should appear.

From the previous discussion it is clear that in order to estimate the spectral and localisation properties of the model of equation (14) we should know how the statistics of the coefficients b_n behaves as a function of n. We evaluated the b_n taking into account lattices with 20201 sites in $d = 2$ and 37881 in $d = 3$. They correspond to 101 iterations of equation (16) in $d = 2$ and 31 iterations in $d = 3$. The calculations were performed using a statistical ensemble which consisted of more than 500 samples. The mean and variance of b_n for a given n were computed for various disorder distributions. The results

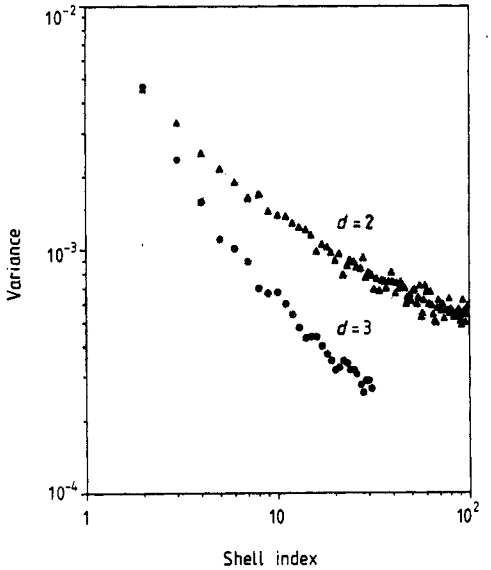

Figure 1. The variance σ_n^2 characterising the distribution of the b_n (equation (14)) is shown for weak off-diagonal disorder plotted against the shell index n for squared ($d = 2$) and cubic ($d = 3$) lattices. The off-diagonal matrix elements V are random variables chosen from a flat probability distribution with zero mean and width of a half.

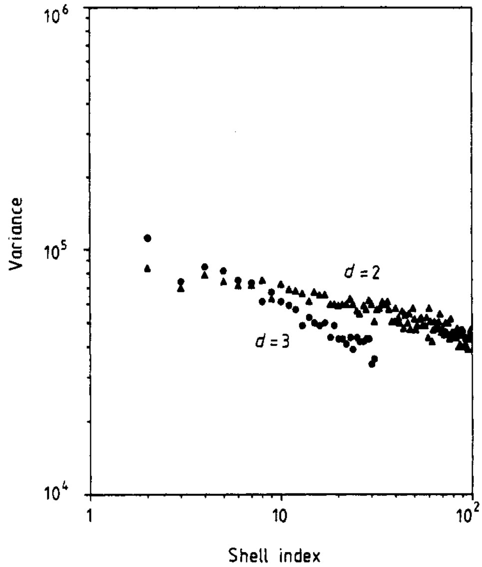

Figure 2. The same as in figure 1 but for strong off-diagonal disorder. The value which characterises the degree of disorder is given by the variance of the box distribution for ln V, $\sigma^2 = 16.0$. It can be seen now that the variance σ_n^2 for the distribution of the b_n, as a function of n, does not decay so rapidly. For even higher values of σ^2 the homogeneous disorder limit should be reached and the b_n become independent.

obtained are shown in figures 1 and 2. The $\langle b_n \rangle$ always converge slowly and their variance decays as a function of n. In figure 1 we plot σ_n^2, the variance of b_n, against the shell index n for weak disorder in a double-log plot. The observed decay could be approximately fitted to a power law with an exponent $\alpha \simeq \frac{1}{4}$ in $d = 2$ and α around $\frac{1}{2}$ for $d = 3$. These results are half the values of those obtained with $\alpha = \frac{1}{2}(d-1)$ which is an estimate for very weak disorder based on simple structural arguments [14] (the number of sites at the nth shell is proportional to n^{d-1}). They can be interpreted as evidence for localisation in $d = 2$, even for weak disorder. Extended states and ordinary diffusion for the classical problem may occur only for $d = 3$ when $\alpha \simeq \frac{1}{2}$. For strong disorder we find that the variances do not decay as rapidly. In figure 2 we show our results in $d = 2, 3$ for some reasonably large values of disorder. The trend of the data is to become independent of n and this is found to be enhanced for even larger values of disorder. If we consider α plotted against σ^2 we observe a rather sharp decay in both $d = 2, 3$ and for large σ^2 the value of $\alpha \simeq 0$ should be recovered asymptotically. Therefore, the known results from section 2 should, asymptotically, describe the properties of strongly disordered systems in any dimension. We should remark that the strengths of the disorder σ^2 required for the one-dimensional description to be valid ($\alpha \simeq 0$) may be unrealistically large to account for real cases. Of course, the higher the dimension the more disordered the system needs to be in order to make the one-dimensional description valid. To conclude, for very strong disorder the one-dimensional behaviour sets in and the $1/|E|$ singularity eventually arises.

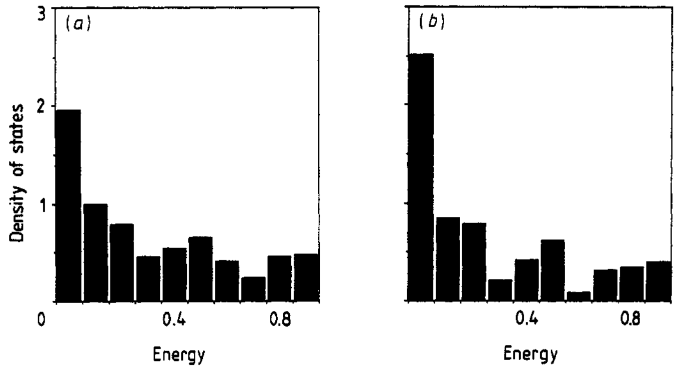

Figure 3. Plot of the averaged normalised density of states $\langle\rho(E)\rangle$ in histogram form in three dimensions for two values of the uncorrelated strong off-diagonal disorder: (*a*) $\sigma^2 = 4.0$; (*b*), $\sigma^2 = 8.0$.

We have also studied this problem via independent means. We computed the averaged integrated density of states $N(E)$ for $|E|$ close to zero in two and three dimensions. We apply a very efficient numerical method [19] which makes use of eigenvalue counting algorithms. The results are displayed in figures 3(*a*) and (*b*) for strong off-diagonal disorder in $d = 3$. The histograms of the averaged density of states are convincing for the existence of a $1/|E|$-type of behaviour even in three dimensions. In figure 4 we have plotted $[N(E) - 0.5]^{-1/2}$ against $1/|E|$ for two reasonably strong values of disorder. For the Dyson singularity to appear $N(E)$ should be logarithmically dependent on $|E|$. We find that for energies close to $|E| = 0$ the behaviour is close to being logarithmic, but it is rather difficult to ensure this when the asymptotic limit is reached.

Finally, we have considered a question which arises naturally in the context of weak off-diagonal disorder [16, 19]. It concerns the upper critical dimension for the existence of the singularity of $\langle\rho(E)\rangle$ at the band centre. Of course, for strong disorder from the present paper it can be concluded that the upper critical dimension should be infinite. From weak-disorder loop expansions [15, 16] it is suggested that it is two. In $d = 3$ only a square root law describes $\langle\rho(E)\rangle$ and no singularity was predicted. We present results for weak disorder in three dimensions in figure 5. The square root law is certainly valid, but only for $|E|$ not too close to zero. A similar situation to the two-dimensional case [19] may occur here where a crossover from the logarithmic to the power-law divergence of equation (15) is seen by lowering $|E|$. In $d = 3$ very close to $|E| = 0$ higher-order powers of $|E|$ become important. We should remark here that it is very difficult to distinguish numerically between very small power laws and logarithmic behaviour but, within reasonable limits, the results obtained in this section allow some definite conclusions to be drawn. Our results do not favour the interpretation of [24] concerning the conclusions reached in [4] as due to the special choice for the distribution of hopping probabilities. The conjecture of [24] concerning ordinary diffusive behaviour in all dimensions is clearly ruled out on the basis of the present results, at least when the disorder is strong.

4. Conclusions

We have shown that Anderson localisation provides a new mechanism for $1/f$ noise which can be displayed in the form of a Dyson spectral singularity. The Dyson result is

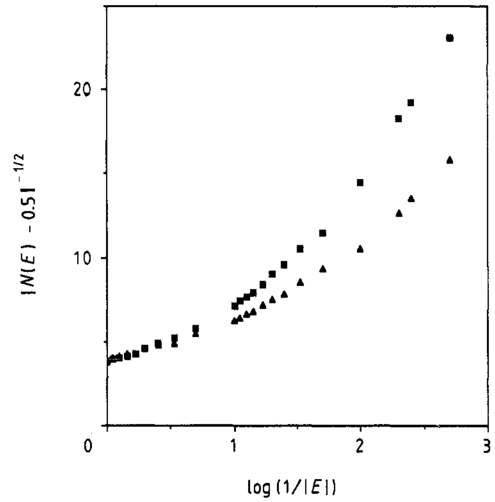

Figure 4. Integrated density of states plotted as $[N(E) - 0.5]^{-1/2}$ against $1/|E|$ on a semi-logarithmic graph for three dimensional $20 \times 20 \times 50$ lattices. The data are collected for rather strong off-diagonal disorder ($\sigma^2 = 4.0$ and 8.0). The Dyson result from equation (4) should be indicated by an asymptotically straight line for large $1/|E|$.

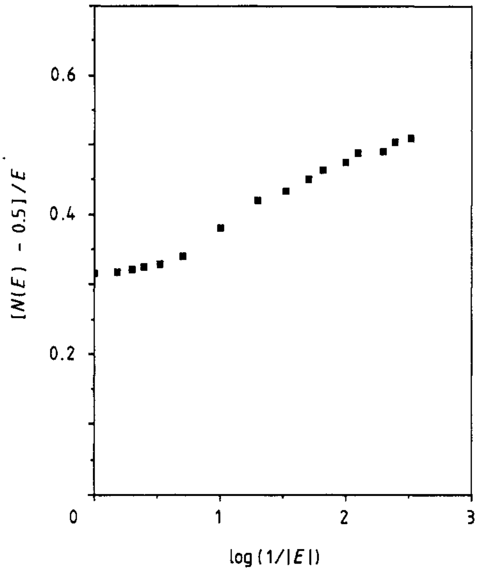

Figure 5. Integrated density of states $[N(E) - 0.5]/|E|$ plotted against $|E|$ in a semi-logarithmic graph for the three-dimensional cubic lattice of size $20 \times 20 \times 50$. The disorder for the V is Gaussian with mean zero and variance of one sixth. Two types of behaviour are distinguished: for not too small $|E|$ (on the left-hand side of the figure) the density of states is $\langle \rho(E) \rangle \simeq c_0 - c_1 \sqrt{|E|} + O(|E|)$, for example, see [16]. For the smaller-$|E|$ region higher-order terms in $|E|$ become important.

simply a manifestation of localisation and the strong sample-to-sample fluctuations in the disordered electronic system and relies on the underlying validity of the log–normal distribution. In the insulating state most ensemble distributions are, indeed, logarithmically normal [8]. They arise because of the inherent multiplicative random processes (transfer matrix products) in the random ensemble. It can be shown that log–normal distributions always lead to $1/f$ noise. The real and imaginary parts of the averaged Green function G give the Lyapunov exponent γ and the averaged integrated density of states $N(E)$, respectively. The strongest possible singularity which is compatible with the integrability of G is a $1/|E|$ singularity. For weaker off-diagonal disorder or correlated disorder different forms of spectral singularities are expected instead. No $1/f$ noise then occurs in the classical analogue.

In this paper we considered a microscopic model of disorder which permits the direct evaluation of its spectral properties and a distinction between weak and strong disorder in any dimension to be made. We are able to link the small-$|E|$ quantum behaviour with the long-time dynamics of models for classical evolution in a random environment and excitation problems. Therefore, a lot of work which has been done in the context of probability theory carries over to quantum problems. For one-dimensional classical problems it is known that anomalous diffusion is usually expected expressed in terms of power laws. For the marginal asymmetric case the Sinai model [5] becomes appropriate, where after a long time t the particle has travelled a mean squared distance proportional only to $\ln^4 t$. These problems require the study of distributions rather than simple averages. For the quantum problem the generalised Lyapunov exponents are defined in terms of the averaged moments of the wavefunction. The $1/|E|$ singularity is the effect of the second cumulant which is the variance of the logarithmic response in equation (9). This is due to the fact that the underlying distributions are log-normal. For weaker disorder the scaling properties of higher-order cumulants must be considered to be often described by an infinite hierarchy of non-related multifractal exponents [12, 13, 25].

In summary, flicker noise and Dyson singularities are closely related. They are both due to untypical events in the statistical ensemble which dominate over the mean values. They should occur for very strong values of disorder, independently of dimension, when the medium is random enough so that it is rich in rare events. The underlying mechanism of this behaviour is the log-normal distribution. These distributions are familiar from relaxational dynamics in complex systems, such as glasses and spin glasses, where $1/f$ phenomena are also common [4, 7]. The mechanism of noise is similar to that in the present study but clearly different from $1/f$ noise due to the phase space structure in the transition to classical chaos. It should be further stressed that although we studied a specific model the underlying logarithmic-normal fluctuations are independent of the kind of disorder. The choice of the model just enables an easy and explicit demonstration of the phenomenon. Many questions in this area still remain mostly concerning the nature of the crossover from power law to Dyson behaviour. In that case the known difficulties [8] with scaling at the Anderson transition may arise. In this paper it is shown that for strong disorder scaling exists and only two parameters are sufficient. Moreover, it should be possible to detect the phenomena dealt with in this paper in experimental studies of fluctuations in insulators. The Hamiltonian with off-diagonal disorder may also serve as a model for current noise in discontinuous metallic films and even flicker noise in biological membranes.

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